

Product Bases in Quantum Information Theory

David P. DiVincenzo and Barbara M. Terhal

ABSTRACT. We review the role of product bases in quantum information theory. We prove two conjectures which were made in [DMS⁺], namely the existence of two sets of bipartite unextendible product bases, in arbitrary dimensions, which are based on a tile construction. We pose some questions related to complete product bases.

1. Introduction

Quantum information theory is concerned with the applications of quantum mechanics in information theory. One of the striking features of quantum mechanics is the capacity of quantum states to be entangled, that is, a pure state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ need not be of the form $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$.

Entanglement has turned out to play a role as a resource in quantum information theory; sharing an entangled state enables parties A and B (Alice and Bob) to transmit quantum information to each other by merely sending classical bits, via the protocol of quantum teleportation [BBC⁺93]. It is however not only quantum entanglement which marks quantum mechanics as a theory with fundamentally different features than classical mechanics. The fact that quantum mechanics is concerned with noncommutative objects is reflected in further applications of quantum mechanics in information theory. An early example is the idea of quantum cryptography [BB84], which is based on the disturbance versus information gain trade-off in quantum states. A more recent development concerns the local distinguishability of sets of mutually orthogonal unentangled (or 'product') states, which we call product bases [BDF⁺99]. The idea is the following.

Given is set S of product states $\{|\alpha_i\rangle \otimes |\beta_i\rangle\}_{i=1}^{|S|}$ in a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The states have the property that they are mutually orthogonal. When any of these states is presented to an Alice and Bob pair with the question which of the states is given, then the Alice and Bob pair will be able to answer the question by simply carrying out a quantum measurement distinguishing the orthogonal states from each other. However, we may constrain Alice and Bob in their actions in such a way that Alice is only allowed to operate on space \mathcal{H}_A and Bob on space \mathcal{H}_B . Furthermore, we do allow them to (classically) communicate their results of measurements and other local actions. Then we ask again; are they able to tell the states in the product bases apart? In [BDF⁺99] it was rigorously proved that, for a certain set of orthogonal product states, the answer is no.

The lack of local distinguishability is tied up with the fact that the projectors on a set of orthogonal product states need not be locally commuting. This feature of nonlocality has been useful in several contexts: it has permitted an extension of Gleason's theorem to multipartite systems[**Wal**]; it is also involved in a class of product bases which were introduced in [**BDM⁺99**], the unextendible product bases (UPB), which we will discuss further here. Let us give the definition of an unextendible product basis in a bipartite Hilbert space (the definition is analogous in multipartite spaces):

DEFINITION 1. *Let \mathcal{H} be a finite dimensional Hilbert space of the form $\mathcal{H}_A \otimes \mathcal{H}_B$. A partial product basis is a set S of mutually orthonormal pure product states spanning a proper subspace \mathcal{H}_S of \mathcal{H} . An unextendible product basis is a partial product basis whose complementary subspace \mathcal{H}_S^\perp contains no product state.*

Unextendible product bases have proved to be extremely rich mathematical objects. It was shown in [**BDM⁺99**] and [**DMS⁺**], that aside from features of local indistinguishability, these product bases relate to the phenomenon of bound entanglement [**HHH98**]. Furthermore, via the connection with bound entanglement, it was shown in [**Ter**] that from every unextendible product basis one can construct an indecomposable positive linear map. Also, in [**AL**] a graph theoretic construction of unextendible product bases with minimal size in arbitrary dimensions and parties was presented. In all, we believe that it would be highly desirable to develop a systematic theory of product bases, unextendible, uncompletable, or complete but 'frustrated', see Section 3.

In this paper we prove two conjectures which were made in [**DMS⁺**], namely the existence of two sets of bipartite unextendible product bases, in arbitrary dimensions. In Section 3 we address some questions related to complete product bases. Let us first recall the definition of these two sets of candidate UPBs.

GenTiles1 is a bipartite product basis in $\mathcal{H}_n \otimes \mathcal{H}_n$ where n is even, where \mathcal{H}_n denotes a n -dimensional Hilbert space. These states have a tile structure which in the case of $\mathcal{H}_6 \otimes \mathcal{H}_6$ is shown in Fig. 1a. The general construction goes as follows: We label a set of n orthonormal states as $|0\rangle, \dots, |n-1\rangle$. We define the set of 'vertical tile' states

$$(1) \quad |V_{mk}\rangle = |k\rangle \otimes |\omega_{m,k+1}\rangle = |k\rangle \otimes \sum_{j=0}^{n/2-1} \omega^{jm} |j+k+1 \bmod n\rangle, \\ m = 1, \dots, n/2-1, \quad k = 0, \dots, n-1,$$

where $\omega = e^{i4\pi/n}$. Similarly, we define the set of 'horizontal tile' states:

$$(2) \quad |H_{mk}\rangle = |\omega_{m,k}\rangle \otimes |k\rangle, \quad m = 1, \dots, n/2-1, \quad k = 0, \dots, n-1.$$

Finally we add a 'stopper' state

$$(3) \quad |F\rangle = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} |i\rangle \otimes |j\rangle.$$

The stopper state is not depicted in Fig. 1; as a tile it would cover the whole 6 by 6 square. The representation of the set as an arrangement of tiles informs us about the orthogonalities among some of its members.

The second set is called **GenTiles2**. It is a construction made in dimensions $\mathcal{H}_m \otimes \mathcal{H}_n$ for $n > 3$, $m \geq 3$ and $n \geq m$. The construction is illustrated in Fig. 2a.

The small tiles which cover two squares are given by

$$(4) \quad |S_j\rangle = \frac{1}{\sqrt{2}}(|j\rangle - |j+1 \bmod m\rangle) \otimes |j\rangle, \quad 0 \leq j \leq m-1.$$

These short tiles are mutually orthogonal on Bob's side. The long tiles (in general not contiguous) that stretch out in the vertical direction in Fig. 2a are given by

$$(5) \quad |L_{jk}\rangle = |j\rangle \otimes \frac{1}{\sqrt{n-2}} \left(\sum_{i=0}^{m-3} \omega^{ik} |i+j+1 \bmod m\rangle + \sum_{i=m-2}^{n-3} \omega^{ik} |i+2\rangle \right),$$

$$0 \leq j \leq m-1, \quad 1 \leq k \leq n-3,$$

with $\omega = e^{i\frac{2\pi}{n-2}}$. Lastly we add a 'stopper' state

$$(6) \quad |F\rangle = \frac{1}{\sqrt{nm}} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |i\rangle \otimes |j\rangle.$$

The total number of states is $mn - 2m + 1$.

2. Two Theorems

THEOREM 1. *The set of states **GenTiles1** forms a UPB on $\mathcal{H}_n \otimes \mathcal{H}_n$ for all even $n \geq 4$.*

PROOF. We proceed by assuming that another product state $|\xi\rangle$ orthogonal to all these exists, and showing that a contradiction results. Expanded in the basis above, $|\xi\rangle$ must have at least one non-zero term, which we will call $|ij\rangle$. That is

$$(7) \quad |\xi\rangle = a|ij\rangle + \dots,$$

with $a \neq 0$. Now, since simultaneous cyclic permutations of the A and B Hilbert space leaves **GenTiles1** invariant, we can always relabel things:

$$(8) \quad |\xi\rangle = a|i-j, 0\rangle + \dots,$$

where $i-j$ is understood to be mod n . If $i-j < n/2$ this is the form we will proceed with. Otherwise, we transform instead to $|\xi\rangle = a|0, j-i\rangle + \dots$, then we interchange A and B with a shift of 1 (another symmetry of this tile set) to obtain $|\xi\rangle = a|j-i-1, 0\rangle + \dots$. In either case the state is written as

$$(9) \quad |\xi\rangle = a|s, 0\rangle + \dots$$

with $0 \leq s \leq n/2 - 1$. Now the real work will consist of narrowing down what the ... can consist of, given that this state must be orthogonal to all the states in the set **GenTiles1**, and must be a product state.

In order for $|\xi\rangle$ to be orthogonal to all the $|H_{m0}\rangle$ states, it must be of the form

$$(10) \quad |\xi\rangle = a \sum_{s=0}^{s=n/2-1} |s, 0\rangle + \dots$$

The state $|\xi\rangle$ must have terms other than the ones shown, since otherwise it could not be orthogonal to $|F\rangle$. We consider two cases: The easy case is

1. *B only has support on $|0\rangle$.* In this case we can write

$$(11) \quad |\xi\rangle = a \sum_{s=0}^{s=n/2-1} |s, 0\rangle + b|t, 0\rangle + \dots$$

for some t , $n/2 \leq t < n$, and some $b \neq 0$. But now $|\langle V_{1t} | \xi \rangle| = |b| \neq 0$. So this case is ruled out; the other case involves considerably more work:

2. *B has support beyond $|0\rangle$.* In this case we can write $|\xi\rangle$ as

$$(12) \quad |\xi\rangle = a \sum_{s=0}^{s=n/2-1} |s, 0\rangle + b |t, r\rangle + \dots$$

for some $r \neq 0$, $b \neq 0$ and $t < n/2$ (if there were only terms with $t \geq n/2$, $|\xi\rangle$ would not be a product state). In fact, since for a product state the A state must be independent of the result of a projection by B onto his basis, the state must therefore have the form

$$(13) \quad |\xi\rangle = a \sum_{s=0}^{s=n/2-1} |s, 0\rangle + b \sum_{s=0}^{s=n/2-1} |s, r\rangle + \dots$$

Fig. 1a gives a graphical depiction of this state for $r = 4$ and $n = 6$. The rest of the argument is easiest to follow using this series of pictures. Fig. 1b shows the additional constraints on the state arising from the orthogonality with the set of states $|V_{m,n/2-1}\rangle$ (if r had been $r < n/2$, we would have used orthogonality with $|V_{m,0}\rangle$ to proceed instead). In symbols, this state is

$$(14) \quad |\xi\rangle = a \sum_{s=0}^{s=n/2-1} |s, 0\rangle + b \sum_{s=0}^{s=n/2-1} |s, r\rangle + b \sum_{\substack{q=n/2, q \neq r \\ q=n-1}}^{q=n-1} |n/2-1, q\rangle + \dots,$$

but at this point it is much easier to understand graphically. Next we again impose the constraint that the state have a product form; graphically this requirement can be explained by saying that all the columns and rows have to be proportional to one another. Thus we “fill out the rectangle” with bs as in Fig. 1c. In symbols, the state now has the description

$$(15) \quad |\xi\rangle = a \sum_{s=0}^{s=n/2-1} |s, 0\rangle + b \sum_{s=0}^{s=n/2-1} \sum_{\substack{q=n/2 \\ q=n-1}}^{s=n-1} |s, q\rangle + \dots$$

We now invoke orthogonality with $|V_{m,0}\rangle$ to fill in the additional bs in Fig. 1d. (We will now dispense with the algebraic expressions altogether.) Then again requiring a product state gives Fig. 1e. To arrive at Fig. 1f we invoke the orthogonality for $|H_{mk}\rangle$, $1 \leq k \leq n-1$, except $k \neq n/2$. Requiring the product form brings us to Fig. 1g. Now by enforcing orthogonality for $|V_{m,n/2}\rangle$ we find that $a = b$, so the entire state is constrained as in Fig. 1h. But this state is just $|F\rangle$, so it fails to be orthogonal to all the states. Therefore, an additional product state does not exist, and **GenTiles1** is a UPB. \square

THEOREM 2. *The set of states **GenTiles2** forms a UPB on $\mathcal{H}_m \otimes \mathcal{H}_n$ for $n > 3$, $m \geq 3$, and $n \geq m$.*

PROOF. We show that this set is a UPB, for all m and n , by the same methods as before. Since **GenTiles2** has less symmetry than **GenTiles1** we will have to examine more cases, but the methods will be the same. We will number the following paragraphs to indicate the structure of the cases being considered.

In all cases we begin by assuming that there is an additional product state $|\xi\rangle$ with nonzero amplitude on some basis state $|ij\rangle$; we will examine all possible values of i and j .

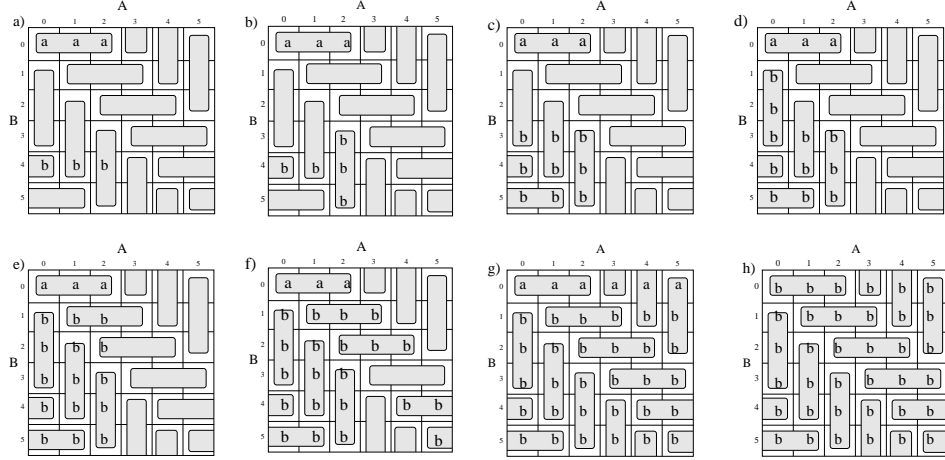


FIGURE 1.

1. Small-tile case. Suppose that $i = j = 0$, so that the state is

$$(16) \quad |\xi\rangle = a|00\rangle + \dots$$

Orthogonality with $|S_0\rangle$ gives

$$(17) \quad |\xi\rangle = a(|00\rangle + |10\rangle) + \dots$$

Note that the outcome would have been the same if we had started with assuming that the amplitude of $|10\rangle$ was nonzero. By relabeling of the Hilbert space, this case will cover any initial $|ij\rangle$ which lies in a small tile.

1a. Consider the case where the B part of the product state is $|0\rangle$. Orthogonality with $|F\rangle$ requires that Eq. (17) have other nonzero terms:

$$(18) \quad |\xi\rangle = a(|00\rangle + |10\rangle) + b|r, 0\rangle + \dots$$

But then $|\langle L_{r1}|\xi\rangle| = |b| \neq 0$, so this case is excluded.

1b. Consider the case where the B part of the product state is not just $|0\rangle$. Then we know that

$$(19) \quad |\xi\rangle = a(|00\rangle + |10\rangle) + b|r, t\rangle + \dots$$

Here r is 0 or 1, and $t > 0$. We now go down to two other subcases, depending on whether the additional term is in a small or a large tile.

1b1. $|r, t\rangle$ is in a small tile (Fig. 2a illustrates the case $r = 0, t = m - 1$). Orthogonality with respect to $|S_{m-1}\rangle$ gives Fig. 2b; then the product constraint, which requires rows 0 and $m - 1$ to be proportional in the example shown, gives Fig. 2c. Orthogonality with respect to $|L_{1k}\rangle$ and $|L_{m-1,k}\rangle$ gives Fig. 2d. The product state condition, which requires rows $m - 2$ and $m - 1$ to be proportional, gives $a = b$ and brings us to Fig. 2e. Orthogonality with $|S_1\rangle$ takes us to Fig. 2f, another application of the product state condition gives Fig. 2g. Finally, repeated application of orthogonality with $|S_i\rangle$, $2 \leq i \leq m - 2$, and the product-state condition, fills in the whole state with b s as in Fig. 2h. But this is just $|F\rangle$, so $|\xi\rangle$ cannot be orthogonal to all of **Gentiles2** in this case.

1b2. $|r, t\rangle$ is in a large tile. Fig. 3a shows the case $r = 0, t = 2$, it will be easy to see that all cases are equivalent. Orthogonality with $|L_{0k}\rangle$ gives Fig. 3b. The

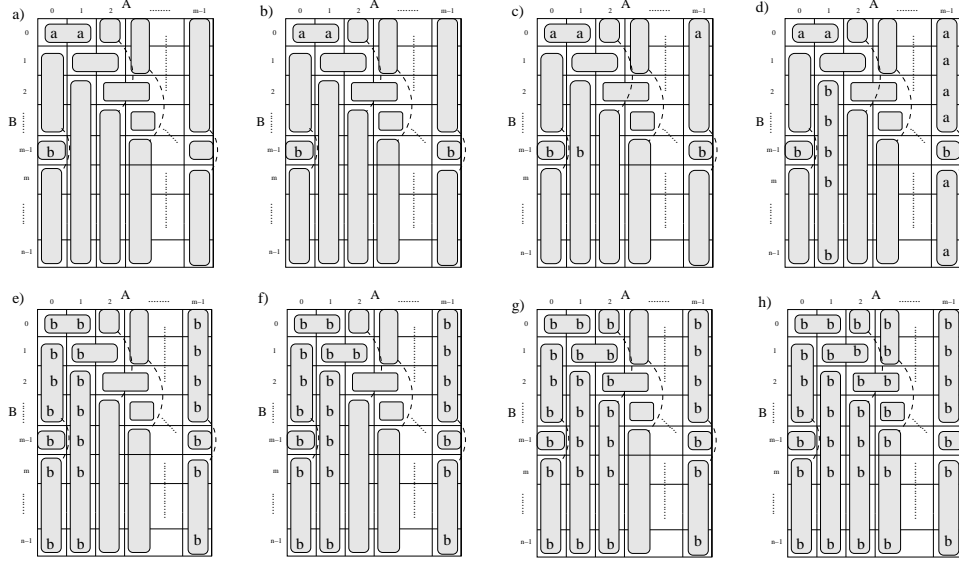


FIGURE 2.

product state condition brings us to Fig. 3c. Orthogonality with $|S_1\rangle$ and $|L_{1k}\rangle$ gives Fig. 3d, and the product state condition gives Fig. 3e. Now, orthogonality with $|L_{2k}\rangle$ requires $a = b$. The remainder of the reasoning follows the same track from Fig. 2g; this case is excluded.

2. Large-tile case. That is, we assume that $|\xi\rangle$ has at least one non-zero element, and it is in a large tile. All large tiles are equivalent by relabeling of the Hilbert space, so we can pick one, say $i = 0, j = 2$. Then by orthogonality to $|L_{0k}\rangle$ we get Fig. 4a. There have to be more non-zero entries since $|\xi\rangle$ must be orthogonal to $|F\rangle$. We consider different subcases corresponding to which other entry is nonzero.

2a. Component in $|S_0\rangle$ or $|S_{m-1}\rangle$. This immediately brings us back to Fig. 3b.

2b. Component in another short tile $|S_r\rangle$. This is illustrated for $r = 2$ in Fig. 4b, using the orthogonality to $|S_r\rangle$. Invoking the product state condition gives Fig. 4c; invoking orthogonality to $|L_{rk}\rangle$ (not shown) brings us back to the situation of Fig. 3b.

2c. Component in another long tile; we illustrate this for $r = 2$, invoking orthogonality to $|L_{rk}\rangle$, in Fig. 4d. The product state condition brings us to Fig. 4e; invoking orthogonality to $|S_0\rangle$ (not shown), we again return to Fig. 3b.

This covers all cases; all methods of constructing the product state $|\xi\rangle$ lead to a contradiction. Thus, **GenTiles2** is a UPB. \square

3. Open questions

Let us consider some questions related to complete product bases, that is, a set of orthonormal product states in some multipartite Hilbert space \mathcal{H} which span the full space. The simplest basis for a bipartite space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is the ‘Cartesian’ basis $\{|i\rangle \otimes |j\rangle\}$, where $\{|i\rangle\}_{i=1}^{\dim \mathcal{H}_A}$ and $\{|j\rangle\}_{j=1}^{\dim \mathcal{H}_B}$ are orthonormal bases for \mathcal{H}_A and \mathcal{H}_B resp. The Cartesian basis has the simple property that the projectors on the local parts of the product states commute. In general one would like to characterize the local noncommutative structure, or the level of ‘frustration’, of a

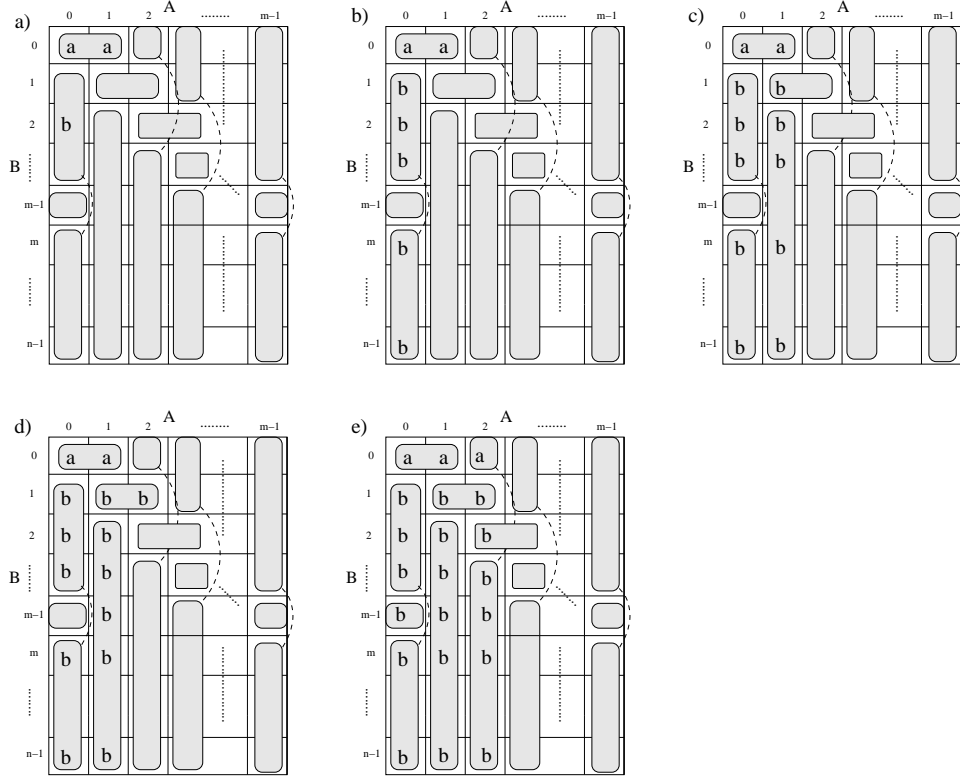


FIGURE 3.

complete product basis; this structure relates to the question of whether members of a product basis for example are distinguishable by local quantum operations and classical communication, see [BDF⁺99]. In order to explore this structure we believe that it is interesting to introduce the following notion of ‘winding’:

DEFINITION 2. Let \mathcal{B} be a Cartesian basis for $\mathcal{H}_A \otimes \mathcal{H}_B$. The procedure of winding is the repeated application of the following two steps:

1. Choose a subspace $\mathcal{H}'_A \otimes \mathcal{H}'_B \subseteq \mathcal{H}_A \otimes \mathcal{H}_B$ such that for every state $|a\rangle \otimes |b\rangle \in \mathcal{B}$ either (1) $|a\rangle \otimes |b\rangle \in \mathcal{H}'_A \otimes \mathcal{H}'_B$ or (2) $|a\rangle \otimes |b\rangle \in (\mathcal{H}'_A \otimes \mathcal{H}'_B)^\perp$.
2. Apply a local unitary transformation $U_A \otimes U_B \in B(\mathcal{H}'_A \otimes \mathcal{H}'_B)$ (only supported on $\mathcal{H}'_A \otimes \mathcal{H}'_B$) on the set \mathcal{B} . Call the new set of partially rotated states \mathcal{B}' .

We refer to the reverse procedure as ‘unwinding’. We note that with this winding procedure $\mathcal{B} \rightarrow \mathcal{B}' \rightarrow \dots \rightarrow \mathcal{B}_{end}$, the states in each \mathcal{B} -set remain orthogonal product states spanning $\mathcal{H}_A \otimes \mathcal{H}_B$. The definition is analogous for complete product bases in multipartite spaces.

By a procedure of winding we can create a complete product basis that has a certain level of local noncommutativity which may be characterized by the number of winding moves. The question whether winding is a good characterization of the local structure of the basis relates to the question whether, say, all bipartite complete product bases can be generated in this manner, i.e.

Question: Are all bipartite complete product bases unwindable?

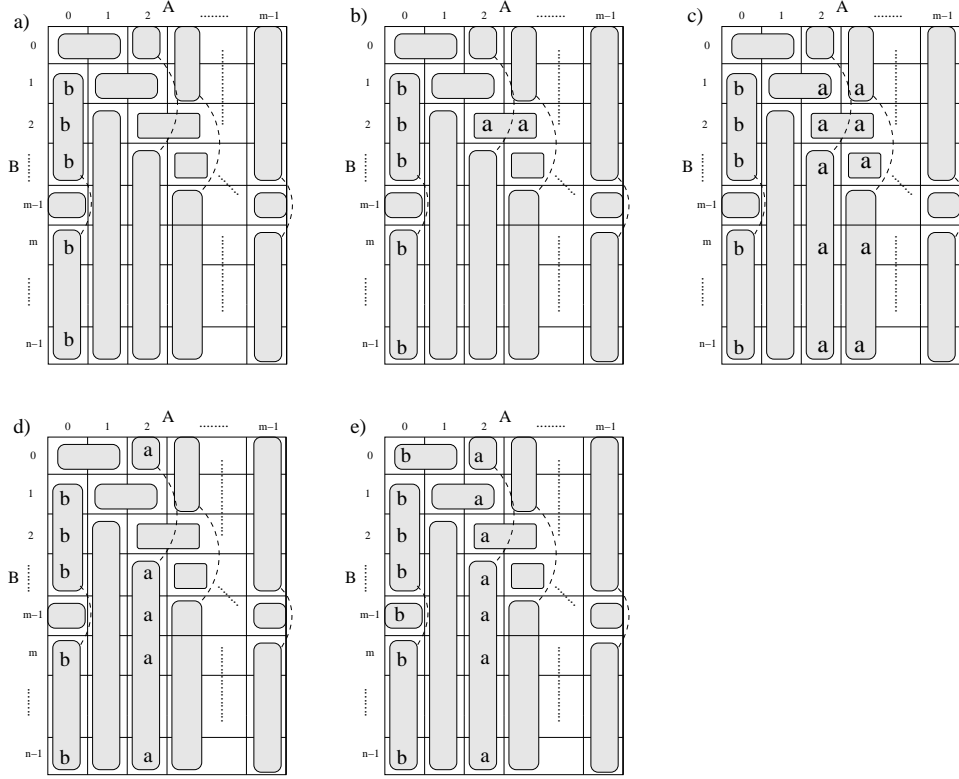


FIGURE 4.

We have not been able to answer this question in general, but we do know that the answer is yes for all complete product bases in $\mathcal{H}_3 \otimes \mathcal{H}_3$, and $\mathcal{H}_2 \otimes \mathcal{H}_n$ for all $n = 2, 3, \dots$. Furthermore we note that it is possible to prove that every complete product basis that is distinguishable by local quantum operations and classical communication [BDF⁺99] is unwindable.

Our question becomes more interesting for multipartite spaces. We have shown that all bases in $\mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_2$ are unwindable. However, there is a surprise: there exists an example of a complete product basis in a 10-partite Hilbert space ($\mathcal{H}_2^{\otimes 10}$) that can be proved to be *not unwindable* [Sho]. This basis emerged from work of Lagarias and Shor [LS92, LS94] that disproved the Keller conjecture of tiling theory in 10 dimensions. In fact, it is the features of this construction that make it a good counterexample to the Keller conjecture that also make it fail to conform to Definition 2: the failure of the ‘face-to-face’ tiling property of Keller corresponds to there being no pair of states confined to a subspace $\mathcal{H}_2 \otimes \mathcal{H}_1^{\otimes 9}$; and the fact that the Lagarias-Shor construction contains no smaller-dimensional counterexamples to the Keller conjecture implies that there are no sets of four states confined to a subspace $\mathcal{H}_2^{\otimes 2} \otimes \mathcal{H}_1^{\otimes 9}$, no sets of eight states confined to $\mathcal{H}_2^{\otimes 3} \otimes \mathcal{H}_1^{\otimes 8}$, etc. These are all the possible subspaces in Definition 2, so no winding moves are possible for this basis. Perhaps further remarkable connections will emerge in the future with tiling theory that will permit further progress to be made on our Question.

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IBM T.J. WATSON RESEARCH CENTER, YORKTOWN HEIGHTS, NY 10598, USA

E-mail address: divince@watson.ibm.com, terhal@watson.ibm.com